On the Ancient Determination of the Meridian Arc Length by Eratosthenes of Kyrene

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Key words:

SUMMARY

For topographic/geographic purposes the size of the Earth is of utmost importance in modern time as well as for Eratosthenes facing the task to draw a map of the "oikumene". His determination of the radius of the Earth resulted in 252000 stadia. If he has used a stadion definition of 1 stadion = 158,7m = 300 Egyptian royal cubits = 600 Gudea units (length of the yardstick at the statue of Gudea (2300 b c) in the Louvre/Paris), he has already observed the meridian arc length to 252000 · 0,1587 = 40000km. How could Eratosthenes obtain in ancient times already such an accurate result?

Ptolemaios describes in his "Geographike hyphegesis" a method used by the "elder" to determine the size of the Earth; this ancient method to measure the meridian arc length between the latitude circles of two cities (e.g. Alexandria/Syene, Syene/Meroe) is based on a traversing technique, as will be shown.

Geographical latitudes could be measured using a "Skiotheron" (shadow seizer). An according to ancient information reconstructed instrument will be shown and explained; the accuracy of sun observations with such a kind of instrument is comparable with those of a modern sextant.

A recovery of the two systems of ancient geographical stadia (Alexandrian and Greek) is presented. It is presently used for a rectification of the digitalised maps given in Ptolemy's "Geographike hyphegesis". The stadion definition Eratosthenes has used (1 meridian degree = 700 stadia) was applied also in northern and western Europe and in Asia east of the Tigris river; using it as a scale factor we got very good results for the rectification.

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1. SOME REMARKS ABOUT THE ANCIENT NON-METRIC SYSTEM OF LENGTH UNITS

As very often stated in the ancient literature Eratosthenes obtained as the result of his determination of the meridian arc length of the earth the figure of 252000 stadia. Some hot debates took place in modern times about the length of the stadion Eratosthenes has been used. The question is related to the definition of the Persian Parasange (= ½ mesopotamian Kaspu) and the Egyptian Schoinos.

Our knowledge about the set of ancient non-metric length units (that are the cubit/foot or pechys/pous units) is mainly based on the Nippur cubit (NC = 518,5mm; museum in Istanbul), the so-called Gudea unit (GU = (20/28)(20/28) NC = 264,55 mm; Louvre / Paris) as well as the pes romanus or “Pous Romaikos” (pr = 16/28 NC = (28/24) GU = 296,3mm). All the ancient metric relations are governed by the very old Egyptian definition of a Remen (Pygon): 1 Remen = (20/28) royal cubits = (20/24) trade cubits = (20/16) “Pous”.

Since about 2000 b.c. a (new) Egyptian Royal cubit (RC = 2 GU =529,1mm) was used in egypt, closely connected to the Gudea unit. If a foot unit used in Rome (palmipes romanus), Milet and china (Tschi) (MF = 6/5 GU = 6/10 RC = 317,5mm) was called by Heron “Pous Philetairikos” (as will be assumed here), then the Gudea unit was called by him “Pous Italikos”. Very probably the “Pous Ptolemaikos” (PP = (2/3) RC = 352,7mm) was connected to the Royal Cubit and to a foot unit used e.g. to rebuild Carthago (PC = (5/6) PP = 293,9mm). Finally, it is well-known, that another foot unit was used in the Kyrenaika (FK = 28/24 GE = 308,6mm), in the homeland of Eratosthenes; the relation Kyrenaika foot: pes romanus = 308,6 : 296,3 = 25 : 24 is well documented in the ancient literature. For more details about the ancient system of non-metric length units see (Lelgemann 2004).

The tables of Senkereh (Lepsius 1877) provide us with the exact definition of the mesopotamian Kaspu (K= 6·60·60 cubits); the Persian Parasange (P = 3·60·60 cubits) was half of a Kaspu. As stated by Herodot the persian Paransange was usually divided into 30 stadia,1 stadion = 600 foot. The question is: What kind of a cubit was used for the definition of the Kaspu?

Because only a very few cubit units have been used in ancient times (in contrast to a large set of foot units) only a very few possibilities remain for the length of the Parasange. By far the most plausible is
1 Parasange = 10800 Royal Cubits = 5,714 km.

As a consequence , for the stadion of Herodot = (1/30) Paransange the corresponding foot unit was the “Pous Philetairikos” (palmipes romanus/Tschi)

1 Stadion of Herodot = 600 Pous Philetairikos = 190,5 m.

Closely connected to the Paransange by a relation 9 : 10 =10800 : 12000 (but also to the Egyptian length units 1 Khet = 100 Royal cubits and 1 Itr = 200 Khet = 20000 Royal cubits) was the Egyptian
Schoinos = 12000 Royal cubits = 6,349 km.

Eratosthenes has used, according to Plinius, a stadion = (1/40) Schoinos, that is

Stadion of Eratosthenes = 300 Royal cubits = 600 Gudea units = 158,7m.

According to Ananias of Schirak (~ 600 AD), in addition to this stadion used for “surveys in the air” another larger stadion, both related to each other by the relation 3 : 4, was used for “surveys on the earth”, that is the

Stadion Ptolemaikos = 400 Royal cubits = 600 Pous Ptolemaikos = 211,6 m.

This was the stadion = (1/30) Schoinos mentioned also by Plinius.

As we know from the roman literature Eratosthenes has used also the

Stadion Italikon = 600 Kyrenaika foot = (28/24) 600 Gudea units = 185,2 m.

That explains also a comment by Strabo, that Eratosthenes has divided the circumference of the earth in “Hexakontades”.
Probably never used by ancient geographers was the

Stadion Olympicon = 600 pes romanus = (28/25) 600 Gudea units = 177,8 m.

In this “Alexandrian stadion system” the meridian arc length could be expressed by:

$$\begin{aligned}\text{Meridian are length} &= 70\cdot 60\cdot 60 \text{ stadia of Eratosthenes} = 60\cdot 60\cdot 60 \text{ Stadia Italicon}\\ &= \tfrac{3}{4}\, 70\cdot 60\cdot 60 \text{ Stadia Ptolemaikos} = 40.000 \text{ km.}\end{aligned}$$

Probably within the Stoa-school another system of stadia was used, which will be called here “Greek stadion system”. Indeed, Plinius mentioned also a stadion = (1/32) schoinos used by “others” (members of the Stoa?), which may be called

“Chaldean” stadion = 375 Royal cubits = 750 Gudea units = 198,4 m.

Closely related to this stadion was a stadion definition used in Rhodos by Poseidonios and Artemidoros

“Rhodian” Stadion = (5/6) 375 Royal cubits = (25/24) Gudea units = 165,3 m

as well as a larger stadion, corresponding by the relation 3 : 4, of the length

“Phönician” Stadion = (10/9) 375 Royal cubits = (25/18) 600 Gudea units = 220,5 m.

For greek geographers using those definitions of a stadion a circumference of the earth of 250.000 Stadion Eratosthenes was much more comfortable, because of the simple relations:

Meridian arc length	=	240 000 “Rhodian” stadia	=	200 000 “Chaldean” stadia
	=	180 000 “Phönician” stadia	=	39 700 km.

Indeed, the member of the Stoa Kleomedes mentioned precisely this figure of 250.000 stadia for the circumference of the earth (He does not say, that Eratosthenes has used this figure, as often casually assumed in the modern literature).

Klaudios Ptolemaios, probably following Marinus of Tyros, used in his “Geographike hyphegesis” the “Phönician” Stadion that is 1° = 500 stadia. In this work, our major source of the state of art of geography in the hellenistic/roman epoch, he published the first digitized map in history by providing the geographic longitudes/latitudes of more than 8000 important towns and land marks of the “oikumene”.

As can easily be recognised this map is extremely distorted in particular due to scale errors. A careful analysis of the positional figures given by Ptolemaios has shown, that only a surprisingly few scale factors are necessary for a sufficient rectification: 500/700, 500/600 and 500/525 (as well as about 500/900 in the very small region of the Peloponnes/Attika). Those empirical scale factors, as derived from the numerical data given by Ptolemaios, can obviously be explained by the different definitions of a stadion: Stadion of Eratosthenes (500/700), Stadion Italikon (500/600) and Stadion Ptolemaikos (500/525). In particular, in very large parts of the “oikumene” the scale factor 500/700 must be used, in all north and Western Europe, in parts of Africa and in all Asia east of the Tigris River.

Obviously, the Stadion of Eratosthenes was much used also by other ancient geographers, as the numerical data handed down by Ptolemaios are telling us. There remains an important question. What kind of geodetic methods have been used by the ancient geographers to obtain such accurate maps and in particular what kind of geodetic methods have been used by Eratosthenes to measure the circumference of the earth with an accuracy obtained again in modern times only at about 1800AD?

2. ON THE ASTRO-GEODETIC METHODS USED BY ERATOSTHENES OF KYRENE

A “methods of the old ones” to measure the circumference of the earth is described in the “Geographike hyphegesis” of Klaudios Ptolemaios; it is based on a spherical modification of a planar method used by the “bematists” or “mensores regios Ptolemai” as well as the roman agrimensores.

The bematists observed distances mainly by counting steps; a trained bematist may achieve an accuracy of 10^{-2} for a single distance. However, the direction between two points on the earth, that means the azimuth, is of course as important as the distance. It could only be obtained at that time, as Klaudios Ptolemaios pointed out in the "Geographike hyphegesis", by astronomical methods. It is well-known that the roman agrimensores separated a distance in a north/south component (called "Cardo") and an east/west component (called "Decimanus"). Several methods to fix the meridian direction using the sun are described in the ancient literature. Having fixed the meridian direction for given distances just one simple proportion provides the Cardo x as well as the Decimanus y. (see fig.1)

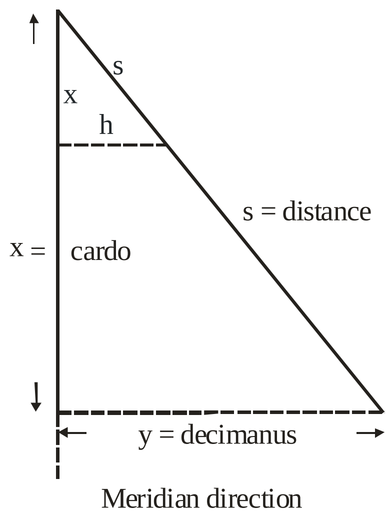

Fig.1: Cardo and Decimanus of a distance in a plane

As explained by Klaudios Ptolemaios, the "method of the old ones" to measure the circumference of the earth was very similar to the method of the roman agrimensores, but instead of using a planar construction it was designed for a sphere based on a most suitable cone projection as outlined in fig. 2 (Knobloch et. al. 2003)

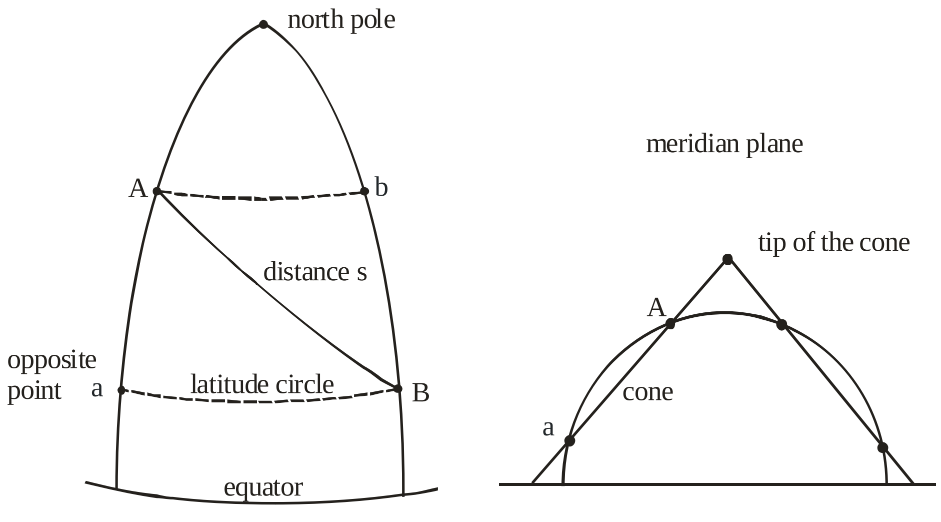

Fig 2: "opposite points" and straight lines through the end points A and B

Establishing a traverse along the Nile-river between Alexandria and Syene/Assuan, by adding the Cardo and Decimanus of the many individual distances between, the length of the meridian arc between the latitude circles of Alexandria and Syene could easily be obtained to 5000 stadia = 793,5km (Times-atlas: 800 km). As we know today, the error propagation of such a method is extremely favourable; it follows a “square root law” such as levelling.

As handed down by Strabo Eratosthenes has estimated the distance between Syene and the Mediterranean to be 5300 stadia. From this figure we can reconstruct the corresponding Decimanus, using $5300^2 - 5000^2 = 1800^2$, to be y = 290 km (Times-atlas: 390 km).

The data agree well also with the figures given in the “Geographike hyphegesis”: Alexandria Φ = 31°00’, Λ = 60°30’, Syene Φ = 23°50’, Λ = 62°00’, Meroe Φ = 16°25’, Λ = 61°30’ (Stevenson 1932).

Of course, the distance between Syene and Alexandria could neither in the past nor today be measured directly, but obviously very easily using the “method of the old ones”. With the same technique the “mensores regios Ptolemai” have been able, as Martianus Capella handed down to us, to measure the length of the meridian arc between the latitude circles of Syene and Meroe (Bagrawia, north of Karthoum); Eratosthenes provided, according to Plinius, again the figure of 5000 stadia = 793,5 km (Times-atlas: 780 km).

Obviously, the traverse Eratosthenes has used for the determination of the circumference of the earth was extended around the tropic of cancer near Syene, 5000 stadia to the north (Alexandria) and 5000 stadia to the south (Meroe).

Kleomedes in a somewhat simplified description of the method assumed Alexandria and Syene to be on the same meridian, but he stated also explicitly that the method Eratosthenes has used was of a more geometrical nature and far more challenging (that means not easily to understand).

Klaudios Ptolemaios mentioned in the “Geographike hyphegesis”, that an instrument called “Skiotheron” (shadow seizer) was used to fix the meridian direction. Our reconstruction of the Skiotheron concept was based mainly on a description Aristophanes has given in his comedy “The birds” for the method the astronomer Meton from Athens has used for sun observations; it resulted in a design as outlined in fig.3.

Mounted movable around the cone on a circular plate, in a horizontal/vertical alignment the tangents of the zenith distance z of the sun can be observed at the second vertical rod as well as the azimuth α of the sun at the circular plate, the circular plate aligned in meridian direction. Tilted by the geographical latitude Φ to the North Pole direction (parallactic alignment) the tangents of the declination δ of the sun can be observed at the second rod as well as the true local solar time τ at the circular plate. In summary, all observable angles (z, ϕ, φ, δ, τ) of the nautical triangle of the sun can be observed easily with such a type of a simple astro-geodetic instrument.

The roman agrimensor Hyginus has described a sophisticated graphical method (analog computer) to fix the meridian direction from any 3 Skiotheron observations of the sun during the day (Lelgemann 2001).

Of course, the Skiotheron can also be used as a kind of scientific “gnomon” to estimate the geographical latitude. This was probably done for the first time by Pytheas of Massalia (Thule expedition); handed down are his very precise “gnomon data” for the latitude of Massalia (Marseille in southern France). The simple idea to measure the circumference of the earth by latitude observations and the measurement of a meridian arc was probably developed long before Eratosthenes.

To derive the geographical latitude ϕ from zenith distances z, the declination δ of the sun and in particular the inclination of the ecliptic ε must be known with high accuracy. As Klaudios Ptolemaios mentioned in the “Almagest” (Mathematike Syntaxis) Eratosthenes has measured this important figure to be 2ε = (11/83) circumference of a circle (that is $\varepsilon = 23°51'$; modern value $\varepsilon = 23°43'$).

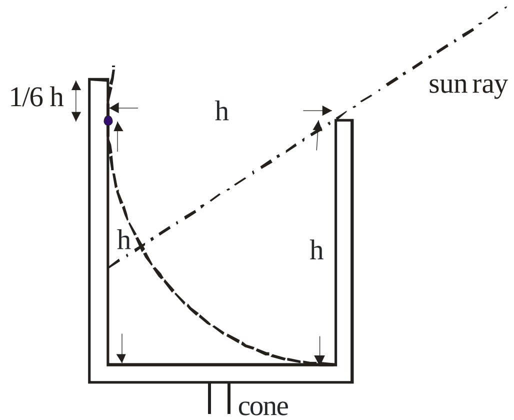

Fig.3: Design of a Skiotheron

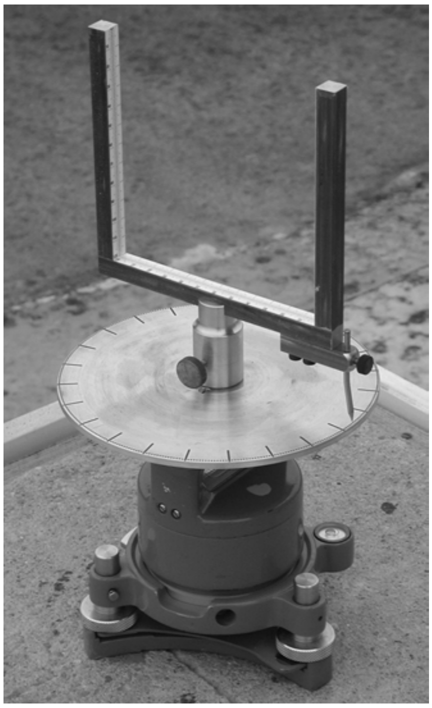

Fig.4: Rebuilding of Skiotheron

Using a modern rebuilding of a Skiotheron (fig.4) it turned out that the accuracy of the observations, after taking into account systematic instrumental errors, are similar to those of a modern sextant, that is a few minutes of arc. Moreover, systematic errors will be nearly the same for two latitude determinations with the same instrument, that is, they vanish with the difference required for grade measurements. Therefore, Eratosthenes had in his hand the method as well as the tools to obtain a precise result for the determination of the circumference of the earth. His very good result is not a case of accidental coincidence, as often claimed in modern times, but of a solid method and very careful observations.

Before the age of satellite geodesy any precise grade measurement was a very expensive task. Therefore, the question remains why the Alexandrians did it at all. Indeed, there was a very practical reason. Eratosthenes (or the library) obtained from Patroklos, a seleucidian general, all information about Asia collected from Alexander the Great, that is, collected from Baiton and Diognetos, the "mensores regios Alexandros". Mainly based on this information Eratosthenes drew a map of the "Oikumene" which was very famous in ancient times.

And for this task, he needed first of all the circumference of the earth as accurate as possible. It is well-known, that Eratosthenes published a report, certainly after intensive studies about the subject, how to measure the size of the earth. This was probably done before starting with the practical surveying. There is nothing mysterious about his very good result; it was the work of a professional just as in modern time. And there is nothing mysterious about the financial and technical support by his scholar, King Ptolemaios: good maps of the "oikumene" were needed for military purposes, trade purposes etc., in the past as well as today.

3. THE TOPOCENTRIC PARALLAX OF THE SUN

Using the Skiotheron technique for astro-geodetic observations one question is concerned with the topocentric parallax, that is, with the difference in the two directions from the topocentre and the centre of mass of the earth to the sun. The topocentric parallax of the moon is fairly large, about one degree of arc. How large was the topocentric parallax of the sun, that was an important question for accurate observations! In order to take the topocentric parallaxe of the sun into account, the distance earth/sun, that is the so called "astronomical unit", must be known or determined, respectively.

As the famous physician and scientist Galenus of Pergamon has handed down from ancient times, Eratosthenes has given also for the astronomical unit (AE = 150 million km) a surprisingly good estimate,

$AE = 804 \text{ Million stadia} \approx 130 \text{ Million km.}$

Neglecting for the moment the question about the method Eratosthenes has used to get such a good result, let us have a look about the implications arising from this particular figure. Since the angle diameter of the sun of about half a degree was well-known, e.g. measured by Archimedes of Syracuse, the size of the sun could easily be estimated, and this size was then extraordinary huge.

If the diameter of the earth is chosen to be 1 cm, the diameter of the sun would be 1 meter and the distance earth/sun about 100 Meter. There are then two possibilities left to explain the "phainomenae".

Either the huge sun revolves the small earth with a fantastic speed once a day (geocentric hypothesis) or the small earth revolves the huge sun with a moderate speed once a year (heliocentric hypothesis).

Of course, from a mechanical point of view there remained for Eratosthenes only one reasonable choice. And therefore, one single number tells us: Eratosthenes was a follower of his teacher Aristarchos of Samos. What was the destiny of the heliocentric concept in ancient times, that is the question!

The work of Eratosthenes leaves no doubt: The hellenistic scientists in Alexandria have based their work on very precise observations and on complex models of applied mathematics/geometry. Therefore, the numbers handed down from ancient times may tell us much more about the state of art of natural sciences in the hellenistic epoch, as generally assumed by modern scientists such as Neugebauer.

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