Efficacy of M_{split} estimation in displacement analysis

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ABSTRACT

Sets of geodetic observations often contain groups of observations which differ from each other in the functional model (or at least in the values of the parameters of such a model). Sets which include the observations from various measurement epochs might be a practical example in such a context. From the conventional point of view, for example in the case of the least squares estimation, subsets in question should be separated before the parameter estimation process. Another option would be an application of M_{split} estimation which is based on a fundamental assumption that each observation is related to several competitive functional models. The optimal automatic assignment of every observation to the respective functional model is one of the objectives of the estimation process. M_{split} estimates of the model parameters are obtained during the iterative process which is based on two (or more) weight functions which also determinate the method properties. Considering deformation analysis, each observation is assigned to the set of the functional models, each of which is related to one measurement epoch. The paper focuses on the efficacy of the method in detecting point displacements. The research is based on example observation sets and with the application of Monte Carlo simulations. The results are compared with the classical deformation analysis, which shows that M_{split} estimation might be an interesting alternative for the conventional methods.

I. INTRODUCTION AND MOTIVATION

Consider the classical functional model of geodetic observations which is given for $l = 1,\ldots,q$ different measurement epochs, namely

$$\mathbf{y} = \mathbf{A}\mathbf{X} + \mathbf{v} \quad \Rightarrow \quad \mathbf{y}_l = \mathbf{A}_l\mathbf{X}_l + \mathbf{v}_l \tag{1}$$

where: $\mathbf{y}_l = [y_{1,l},\cdots,y_{n_l,l}]^T$ are observation vectors which elements belong to the respective sets $\Phi_l = \{y_{1,l},\ldots,y_{n_l,l}\}$; $\mathbf{X}_l = [X_{1,l},\cdots,X_{r,l}]^T$ are parameter vectors; $\mathbf{v}_l = [v_{1,l},\cdots,v_{n_l,l}]^T$ are vectors of random errors, and $\mathbf{A}_l \in R^{n_l,r}$ are known coefficient matrices. Such models are the basis for deformation analysis, namely for determining the shifts $\Delta\mathbf{X}_{(k,l)} = \mathbf{X}_l - \mathbf{X}_k$ between the epochs *l* and *k* (for example, the changes of the point coordinates between such epochs).

The vectors $\Delta\mathbf{X}_{(k,l)}$ can be estimated by applying different methods or strategies (e.g., Pelzer, 1971; Caspary *et al*., 1990; Hekimoglu *et al*., 2010). The least squares method (LS-method) is still the most popular approach in such an analysis, note that LS-estimates are often supplemented with respective statistical tests (e.g., Niemeier, 1981; Setan and Singh, 2001; Denli and Deniz, 2003). However, some unconventional methods are also in use, for example, robust M-estimation (Chen, 1983; Caspary and Borutta, 1987) or R-estimation (Duchnowski, 2010, 2013; Duchnowski and Wiśniewski, 2014, 2017a, 2017b; Wyszkowska and Duchnowski, 2017). In the case of relative networks, one can also apply methods of free adjustment (e.g., Erdogan and Hekimoglu, 2014; Nowel and Kamiński, 2014; Nowel, 2015; Amiri-Simkooei *et al.*, 2017). Some methods as well as their properties are well known, other methods are still being researched. M_{split} estimation surely belongs to the latter group. It was proposed by Wiśniewski (2009, 2010) and it was applied to some practical problems in which each observation could be assigned to several different functional models. For example, it was used in Laser Scanning Data Modeling (Janowski and Rapiński, 2013), deformation analysis (Duchnowski and Wiśniewski, 2012; Zienkiewicz, 2014; Wiśniewski and Zienkiewicz, 2016; Velsink, 2018) or robust estimation (e.g., Li *et al*., 2013). Automatic assignment of each observation to the best fitted model is one of the most important features of M_{split} estimation. It is also very useful in deformation analysis, when the observation set might include observations from all measurement epochs (the set might be unrecognized mixture of such observations). Note, that it is usually no problem with separating observations from different epochs and hence with separate analyses. However, there are some cases when application of M_{split} estimation is advisable. For example, when a point displaces during an observation session, thus, one should consider two pseudo-epochs, and M_{split} estimation allows us to estimate the parameters of the functional models for such pseudo-epochs. Such models can also be applied when an observation set is disturbed by outliers (Zienkiewicz, 2014; Wiśniewski and Zienkiewicz, 2016).

The main properties of M_{split} estimation are discussed in the papers cited above; that paper is focused on the efficacy of the method in estimating parameters of the competitive functional models, hence also estimating point displacements. The analyses are based on simulations of Crude Monte Carlo method and application of elementary functional models or models of a leveling network. The results are compared with the results of LS-method.

II. Theoretical foundations

Without loss of generality we can assume two measurement epochs, thus in the model of Eq. (1) we have $q=2$. Then, the optimization criterion of LS-method and its solution can be written in the following way ($l=1,2$)

$$\begin{aligned}
&\varphi(\mathbf{X}_l)=\sum_{i=1}^{n} v_{i,l}^2 p_{i,l}=\mathbf{v}_l^T\mathbf{P}_l\mathbf{v}_l=\min\\
&\hat{\mathbf{X}}_{LS,l}=\mathbf{D}_{LS,l}\mathbf{y}_l\\
&\mathbf{D}_{LS,l}=(\mathbf{A}_l^T\mathbf{P}_l\mathbf{A}_l)^{-1}\mathbf{A}_l^T\mathbf{P}_l
\end{aligned} \tag{2}$$

where: $\mathbf{P}_l$ are respective weight matrices. The difference $\Delta\hat{\mathbf{X}}_{LS(1,2)}=\hat{\mathbf{X}}_{LS,2}-\hat{\mathbf{X}}_{LS,1}$ is a LS-estimate of the shift $\Delta\mathbf{X}_{(1,2)}$.

In the case of M_{split} estimation, we assume that each observation belongs to either of two sets Φ_1 or Φ_2; however, there is one observation set $\Phi=\Phi_1\cup\Phi_2$ and one observation vector $\mathbf{y}\in R^n$, $n=n_1+n_2$. There are two competitive functional models

$$\mathbf{y}=\mathbf{A}\mathbf{X}_{(1)}+\mathbf{v}_{(1)},\quad \mathbf{y}=\mathbf{A}\mathbf{X}_{(2)}+\mathbf{v}_{(2)} \tag{3}$$

with two competitive versions of the parameter $\mathbf{X}$, namely $\mathbf{X}_{(1)}$ and $\mathbf{X}_{(2)}$ ($\mathbf{A}\in R^{n,r}$, $rank(\mathbf{A})=r$). The vectors $\mathbf{v}_{(1)},\mathbf{v}_{(2)}\in R^n$ are two competitive versions of the observation errors related to all elements of the vector $\mathbf{y}$. Note, that observations can be ordered within the vector $\mathbf{y}$ in any way; however, we can usually assume that $\mathbf{y}=[\mathbf{y}_1^T,\mathbf{y}_2^T]^T$ and hence $\mathbf{A}=[\mathbf{A}_1^T,\mathbf{A}_2^T]^T$. The similar assumptions can also be found in other estimation problems, for example cluster analysis (e.g., Sebert *et al.*, 1998; Soto *et al.*, 2007); or in a mixture model estimation applied in geosciences (e.g., Spurr, 1981; Hsu *et al.*, 1986). Such approaches can be regarded as alternative ones; however, we should have some understanding that they differ much in their general ideas.

The optimization criterion for the squared M_{split} estimation can be written in the following way (Wiśniewski, 2009; Zienkiewicz, 2018)

$$\varphi(\mathbf{X}_{(1)},\mathbf{X}_{(2)})=(\mathbf{v}_{(1)}*\mathbf{v}_{(1)})^T\mathbf{P}^2(\mathbf{v}_{(2)}*\mathbf{v}_{(2)})=\min \tag{4}$$

where: $\mathbf{P}=\mathrm{Diag}(\mathbf{P}_{(1)},\mathbf{P}_{(2)})$ is a combined weight matrix, $*$ is the Hadamard product. The solution can be found by zeroing the respective gradients $\mathbf{g}_{(l)}$ of that function. Generally, for $l=1,2$, one can write

$$\begin{cases}
\mathbf{g}_{(l)}(\hat{\mathbf{X}}_{(l)},\hat{\mathbf{X}}_{(k\neq l)})=2\mathbf{A}^T\mathbf{w}(\hat{\mathbf{v}}_{(k\neq l)})\hat{\mathbf{v}}_{(l)}=\mathbf{0}\\
\mathbf{w}(\hat{\mathbf{v}}_{(l)})=\mathrm{Diag}\left(\hat{v}_{1(l)}^2,\ldots,\hat{v}_{n(l)}^2\right)\\
\boxed{\hat{\mathbf{X}}_{(l)}=\mathbf{D}(\hat{\mathbf{v}}_{(k\neq l)})\mathbf{y}}\\
\mathbf{D}(\hat{\mathbf{v}}_{(l)})=\{\mathbf{A}^T\mathbf{W}(\hat{\mathbf{v}}_{(k\neq l)})\mathbf{A}\}^{-1}\mathbf{A}^T\mathbf{W}(\hat{\mathbf{v}}_{(k\neq l)})\\
\mathbf{W}(\hat{\mathbf{v}}_{(l)})=\mathbf{w}(\hat{\mathbf{v}}_{(l)})\mathbf{P}^2
\end{cases} \tag{5}$$

Since those estimates are functions of the both competitive vectors $\hat{\mathbf{v}}_{(2)}$ and $\hat{\mathbf{v}}_{(1)}$, then such a solution is asymptotic. The following iterative procedure can be applied to compute the sought estimates ($j=1,\ldots,m$)

$$\begin{aligned}
\mathbf{X}_{(1)}^{j+1}&=\mathbf{D}(v_{(2)}^{j})\mathbf{y},\ \ \mathbf{v}_{(1)}^{j+1}=\mathbf{y}-\mathbf{A}\mathbf{X}_{(1)}^{j+1}\\
\mathbf{X}_{(2)}^{j+1}&=\mathbf{D}(v_{(1)}^{j+1})\mathbf{y},\ \ \mathbf{v}_{(2)}^{j+1}=\mathbf{y}-\mathbf{A}\mathbf{X}_{(2)}^{j+1}
\end{aligned} \tag{6}$$

The process stops when for each $l=1,2$ it holds that $\mathbf{g}_{(l)}(\hat{\mathbf{X}}_{(1)},\hat{\mathbf{X}}_{(2)})=\mathbf{0}$ and hence $\hat{\mathbf{X}}_{(l)}=\mathbf{X}_{(l)}^{m}=\mathbf{X}_{(l)}^{m-1}$. Note, that another iterative process which uses both the gradients and the Hessians of $\varphi(\mathbf{X}_{(1)},\mathbf{X}_{(2)})$, namely Newton's method, can be found in (Wiśniewski, 2009, 2010). Here, the shift $\Delta\mathbf{X}_{(1,2)}$ can be estimated by $\Delta\hat{\mathbf{X}}_{(1,2)}=\hat{\mathbf{X}}_{(2)}-\hat{\mathbf{X}}_{(1)}$. It is worth noting that $\Delta\mathbf{X}_{(1,2)}$ can also be estimated directly by applying Shift-M_{split} estimation proposed by Duchnowski and Wiśniewski (2012).

III. Empirical analyses

A. *Elementary tests*

The elementary analysis is based on the univariate models and simulations of observations related to such models. Thus,

$$\begin{aligned}
y_{i,1}&=X_1+v_{i,1},\quad i=1,\ldots,n_1\\
y_{i,2}&=X_2+v_{i,2},\quad i=1,\ldots,n_2
\end{aligned}
\Leftrightarrow
\begin{aligned}
\mathbf{y}_1&=\mathbf{1}_{n_1}X_1+\mathbf{v}_1\\
\mathbf{y}_2&=\mathbf{1}_{n_2}X_2+\mathbf{v}_2
\end{aligned} \tag{7}$$

where: $\mathbf{1}_{n_l} = [1_1, \cdots, 1_{n_l}]^T$; X_1 and X_2 are parameters that differ from each other in the shift $\Delta X_{(1,2)} = X_2 - X_1$. The measurements, namely the elements of the vectors $\mathbf{y}_1$ and $\mathbf{y}_2$, are simulated by using the Gaussian generator $randn(n,1)$ of Matlab. We assume that $\sigma = 1$, and the following theoretical values of the parameters: $X_1^t = 0$ and hence $X_2^t = X_1^t + \Delta X_{(1,2)} = \Delta X_{(1,2)}$. Considering LS-estimation of X_1 and X_2 we can apply the model of Eq. (7) or Eq. (1) where $\mathbf{A}_1 = \mathbf{1}_{n_1}$ and $\mathbf{A}_2 = \mathbf{1}_{n_2}$. In the case of M_{split} estimation we assume the model of Eq. (3) taking $\mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T]^T \in R^n$, $n = n_1 + n_2$ and $\mathbf{A} = \mathbf{1}_n$. We also apply the iterative procedure of Eq. (6) taking LS-estimates as the starting point (note, that the starting point can usually be arbitrary).

Let us now consider one example of observation simulation for which $\Delta X_{(1,2)} = 5\sigma = 5$ and $n_1 = 50$, $n_2 = 10$. The parameter estimates together with the respective residuals are presented in Fig. 1.

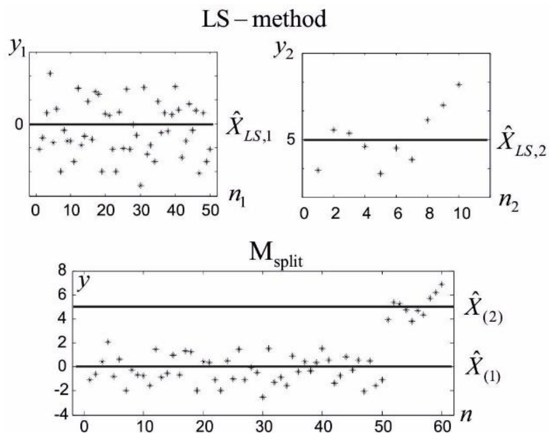

Figure 1. LS and M_{split} estimates for example observation set

Now, let us consider more simulated observation sets. By applying Crude Monte Carlo method (MC) for *N* simulations one can compute MC estimates by applying the formula

$$\hat{\theta}^{MC} = \frac{1}{N} \sum_{i=1}^{N} \hat{\theta}^i \tag{8}$$

where: $\hat{\theta}^i$ are the estimates obtained for *i*th simulation. The location of MC estimates for $N = 5000$ and $\Delta X_{(1,2)} = 5$ or $\Delta X_{(1,2)} = 20$ is presented in Fig. 2. It shows that MC estimates which are obtained for both estimation methods are close to the respective theoretical values (considering the simulated standard deviation). Generally, LS estimates seem more satisfactory. Please note that results obtained for different values of the shift $\Delta X_{(1,2)}$ indicate that M_{split} estimation is more satisfactory for bigger shift than for the smaller one. Thus, let us examine how efficient M_{split} estimation is for different shifts. Let the measure of efficacy be defined in relation to LS estimates, thus let

$$\lambda_{(l)}(\hat{X}_{(l)}, \hat{X}_{LS,l}) = abs(\hat{X}_{(l)} - X_l^t) - abs(\hat{X}_{LS,l} - X_l^t) \tag{9}$$

Note that, when $\lambda_{(l)}(\hat{X}_{(l)}, \hat{X}_{LS,l}) < 0$, then M_{split} estimate is closer to the theoretical value than LS estimate. Now, we can define the following function of an elementary success of M_{split} estimation

$$s_{(l)}(\hat{X}_{(l)}, \hat{X}_{LS,l}) = \begin{cases} 1 & for \;\; \lambda(\hat{X}_{(l)}, \hat{X}_{LS,l}) < 0 \\ 0 & for \;\; \lambda(\hat{X}_{(l)}, \hat{X}_{LS,l}) > 0 \end{cases} \tag{10}$$

Application of MC simulations allows us to present the success rate (SR), which can be computed for different values of the shift $\Delta X_{(1,2)}$

$$\gamma_{(l)}(\hat{X}_{(l)}, \hat{X}_{LS,l}) = \frac{1}{N} \sum_{i=1}^{N} s_{(l)}^i(\hat{X}_{(l)}, \hat{X}_{LS,l}) \tag{11}$$

where: $s^i(\hat{X}_{(l)}, \hat{X}_{LS,l})$ is the value of Eq. (10) at *i*th simulation.

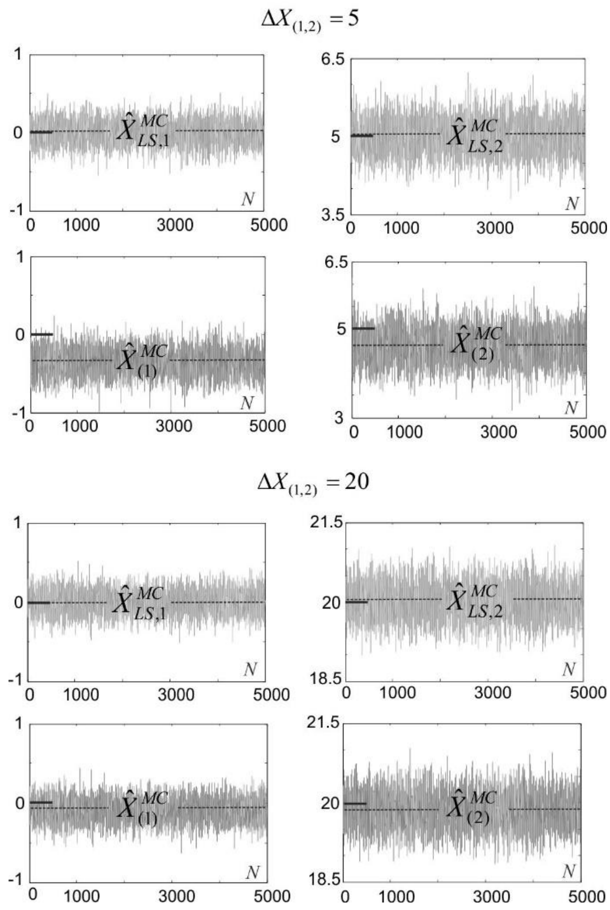

Figure 2. Location of MC estimates for $\Delta X_{(1,2)} = 5$ or $\Delta X_{(1,2)} = 20$

Note, that such an SR is defined in a very similar way to the mean success rate (MSR) given by Hekimoglu and Koch (2000). SRs for different $\Delta X_{(1,2)}$ and for $N = 5000$ simulations are presented in Fig. 3.

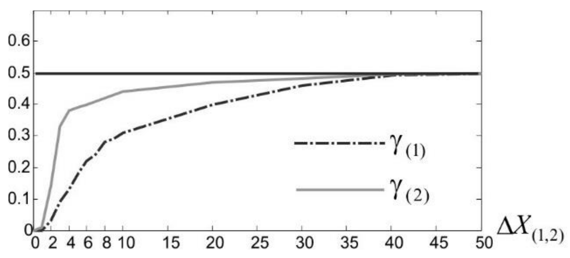

Figure 3. SR of M$_{split}$ estimates $\hat{X}_{(1)}$ and $\hat{X}_{(2)}$ for growing value of $\Delta X_{(1,2)}$

B. *Vertical displacement analysis*

Let us now consider the efficacy of M$_{split}$ estimates in more practical example, namely analysis of vertical displacements within the leveling network which is presented in Fig. 4. Such a network has already been under investigation in some previous papers (Wiśniewski and Zienkiewicz, 2016; Velsink, 2018).

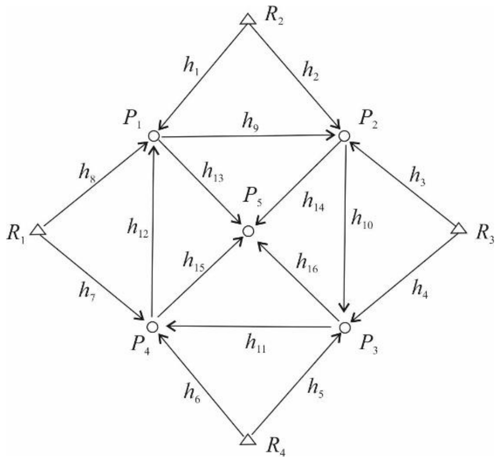

Figure 4. Tested leveling network

The network consists of four reference points $R_1, \ldots, R_4$ with the known heights $H_{R_1} = \cdots = H_{R_4} = 0$ m and five object points $P_1, \ldots, P_5$. We assume that each of the height differences $h_1, \ldots, h_{16}$ is measured twice at each of two measurement epochs, and $\sigma = 2$ mm is the known standard deviation of all measurements. We also assume that at the first epoch $\mathbf{X}_1^t = [H_{1,1} = 0, \cdots, H_{5,1} = 0]^T = \mathbf{0}$, where: $H_{i,1}$ is a height of the *i*th object point at the first epoch. The shift of the object points between the measurment epochs is given by $\Delta\mathbf{X}_{(1,2)} = [\Delta H_{1(1,2)}, \cdots, \Delta H_{5(1,2)}]^T$ where: $\Delta H_{i(1,2)} = H_{i,2} - H_{i,1}$. In the classical approach to estimation of the point displacements, we use the functional model of Eq. (1). Since all height differences are measured twice at two measurement epochs, namely we have two series of measurements at each epoch, then we should assume that $\mathbf{y}_l \in R^{32}$, $\mathbf{X}_l = [H_{1,l}, \cdots, H_{5,l}]^T$, and $\mathbf{A}_{\otimes} = \mathbf{A} \otimes \mathbf{1}_2 \in R^{32,5}$ where: $\mathbf{A} \in R^{16,5}$ is a known coefficient matrix related to one series of measurements, $\mathbf{1}_2 = [1,1]^T$ and $\otimes$ is the Kronecker product. On the other hand, in the case of M$_{split}$ estimation we should apply the functional model of Eq. (3) for which we have $\mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T]^T \in R^{64}$, $\mathbf{A} = [\mathbf{A}_{\otimes}^T, \mathbf{A}_{\otimes}^T]^T \in R^{64,5}$, and $\mathbf{X}_{(1)}, \mathbf{X}_{(2)} \in R^5$ are the competitive versions of the parameter vector, hence $\mathbf{v}_{(1)}, \mathbf{v}_{(2)} \in R^{64}$ are the respective competitive versions of the measurement errors.

When analyzing the efficacy of M$_{split}$ estimation we can use two measures, namely the local measure of the distance between LS and M$_{split}$ estimates

$$\begin{aligned}\lambda_{(l)j}([\hat{\mathbf{X}}_{(l)}]_j, [\hat{\mathbf{X}}_{LS,l}]_j) &= \\ &= abs([\hat{\mathbf{X}}_{(l)}]_j - [\mathbf{X}_l^t]_j) - abs([\hat{\mathbf{X}}_{LS,l}]_j - [\mathbf{X}_l^t]_j)\end{aligned} \quad (12)$$

as well as the global one

$$\lambda_{(l)}(\hat{\mathbf{X}}_{(l)}, \hat{\mathbf{X}}_{LS,l}) = \left\|\hat{\mathbf{X}}_{(l)} - \mathbf{X}_l^t\right\| - \left\|\hat{\mathbf{X}}_{LS,l} - \mathbf{X}_l^t\right\| \quad (13)$$

where: $[\bullet]_j$ is *j*th element of the vector, and $\|\bullet\|$ is the Euclidean norm. The local distance, which is just another form of Eq. (9), is related to a particular parameter, for example, a height of a displacing point. The global distance describes the whole parameter vector. Thus, we can define the local and global success rates in the following way

$$\begin{aligned}\gamma_{(l),j}([\hat{\mathbf{X}}_{(l)}]_j, [\hat{\mathbf{X}}_{LS,l}]_j) &= \frac{1}{N}\sum_{i=1}^{N} s_{(l),j}^i([\hat{\mathbf{X}}_{(l)}]_j, [\hat{\mathbf{X}}_{LS,l}]_j) \\ \gamma_{(l)}(\hat{\mathbf{X}}_{(l)}, \hat{\mathbf{X}}_{LS,l}) &= \frac{1}{N}\sum_{i=1}^{N} s_{(l)}^i(\hat{\mathbf{X}}_{(l)}, \hat{\mathbf{X}}_{LS,l})\end{aligned} \quad (14)$$

where: $s_{(l)j}^i([\hat{\mathbf{X}}_{(l)}]_j, [\hat{\mathbf{X}}_{LS,l}]_j)$ and $s_{(l)}^i(\hat{\mathbf{X}}_{(l)}, \hat{\mathbf{X}}_{LS,l})$ are functions of an elementary success from Eq. (10) and indexed with the respective arguments.

The empirical analysis, which is based on MC method for $N = 5000$ simulations, is carried out for several variants of the point displacements. Firstly, we assume that only the point P_5 is displaced. The respective MC estimates obtained for LS and M$_{split}$ estimations and $\Delta H_{5(1,2)} = -50$, $\Delta H_{5(1,2)} = -100$ or $\Delta H_{5(1,2)} = -200$ mm are presented in Table 1 which also presents the local and global SRs.

MC estimates are similar for both estimation methods and for stable points. SRs indicate that LS estimates are closer to the theoretical values in the vast majority of

Table 1. MC estimates of the point heights and SRs for one unstable point

$\Delta H_{5(1,2)} = -50$				$\Delta H_{5(1,2)} = -100$				$\Delta H_{5(1,2)} = -200$			
$\hat{\mathbf{X}}_{LS,1}$	$\hat{\mathbf{X}}_{(1)}$	$\hat{\mathbf{X}}_{LS,2}$	$\hat{\mathbf{X}}_{(2)}$	$\hat{\mathbf{X}}_{LS,1}$	$\hat{\mathbf{X}}_{(1)}$	$\hat{\mathbf{X}}_{LS,2}$	$\hat{\mathbf{X}}_{(2)}$	$\hat{\mathbf{X}}_{LS,1}$	$\hat{\mathbf{X}}_{(1)}$	$\hat{\mathbf{X}}_{LS,2}$	$\hat{\mathbf{X}}_{(2)}$
0.2	-3.1	0.4	2.9	-0.5	-1.1	-0.6	0.6	0.9	-1.8	-0.7	-0.4
1.4	-1.2	-1.0	0.9	-0.4	-1.3	0.7	2.9	0.5	0.7	0.5	-0.8
2.1	-0.6	-0.6	-0.6	0.1	-0.6	0.5	1.3	-0.3	-2.8	-0.1	1.1
-0.8	-3.6	-0.6	0.6	1.1	-0.9	-1.0	1.2	0.0	-0.4	-1.5	1.5
0.8	-1.9	-50.4	-49.1	0.3	-1.2	-99.8	-98.7	0.8	-0.8	-200.1	-199.7
$\gamma_{(1)} = 0.018$ $\gamma_{(1),5} = 0.172$		$\gamma_{(2)} = 0.019$ $\gamma_{(2),5} = 0.182$		$\gamma_{(1)} = 0.020$ $\gamma_{(1),5} = 0.177$		$\gamma_{(2)} = 0.017$ $\gamma_{(2),5} = 0.187$		$\gamma_{(1)} = 0.025$ $\gamma_{(1),5} = 0.171$		$\gamma_{(2)} = 0.024$ $\gamma_{(2),5} = 0.196$	

the simulations. Note, that the local SRs which are obtained for the point P_5 are much higher than the global ones.

In the second variant, we assume that there are two unstable points, namely P_5 and P_4. The results, which are obtained for the different point shifts, are presented in Table 2. MC estimates obtained for both methods are also similar, here. Fig. 5 presents the LS and M$_{split}$ estimates which are obtained for all MC simulations. Generally, it confirms correctness of both estimation methods; however, differences between those two estimation methods are also apparent. The main difference is the dispersion which is bigger in the case of M$_{split}$ estimation, especially for the stable points, which suggest that the accuracy of M$_{split}$ estimation is worse than LS estimation. It is also worth noting that SRs of M$_{split}$ estimation achieve bigger values in this variant. In the case of the point P_5, the results of M$_{split}$ estimation are better than the results of the classical approach in almost one third of simulations.

The results, which are presented here, show that both methods, namely LS and M$_{split}$ estimation, yield satisfactory solutions. However, such a conclusion is valid for the ordered observation sets, namely when each observation is properly assigned to its measurement epoch. If such a condition is not met,

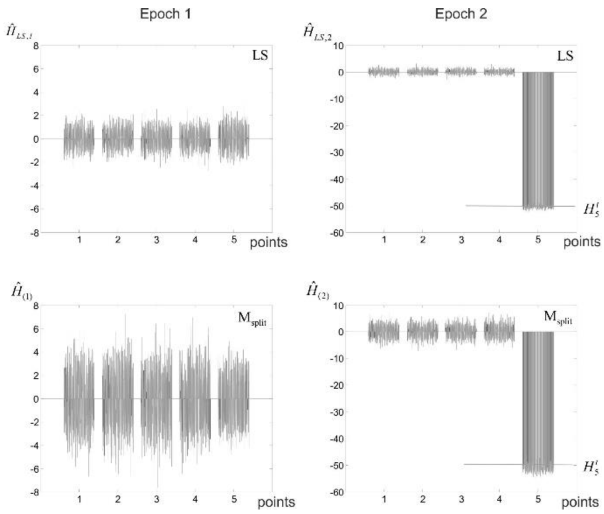

Figure 5. LS and M$_{split}$ estimates of MC simulations ($\Delta H_{4(1,2)} = 50$ and $\Delta H_{5,(1,2)} = 100$ mm)

then the observation from another epoch will usually be regarded as an outlier. Since LS estimation as well as M$_{split}$ estimation are not robust against outliers, they both break down (please note, that M$_{split}$ estimation is generally not robust unless we introduce an additional virtual model for outliers). Note, that in the context addressed here, the outliers result from the assignment of an observation to the wrong measurement epoch

Table 2. MC estimates of the point heights and SRs for two unstable points

$\Delta H_{5(1,2)} = -50;\ \Delta H_{4(1,2)} = -50$				$\Delta H_{5(1,2)} = -100;\ \Delta H_{4(1,2)} = -50$				$\Delta H_{5(1,2)} = -200;\ \Delta H_{4(1,2)} = -50$			
$\hat{\mathbf{X}}_{LS,1}$	$\hat{\mathbf{X}}_{(1)}$	$\hat{\mathbf{X}}_{LS,2}$	$\hat{\mathbf{X}}_{(2)}$	$\hat{\mathbf{X}}_{LS,1}$	$\hat{\mathbf{X}}_{(1)}$	$\hat{\mathbf{X}}_{LS,2}$	$\hat{\mathbf{X}}_{(2)}$	$\hat{\mathbf{X}}_{LS,1}$	$\hat{\mathbf{X}}_{(1)}$	$\hat{\mathbf{X}}_{LS,2}$	$\hat{\mathbf{X}}_{(2)}$
-0.2	-0.5	-0.3	0.6	-0.5	0.1	-0.4	0.3	0.0	-1.3	-0.3	-1.1
-0.4	-2.0	-0.1	-0.1	0.2	0.5	-0.2	-0.4	0.0	-0.2	-0.1	0.9
-0.4	-0.4	-0.9	0.2	0.1	-0.3	0.2	-0.3	-0.2	-0.2	-0.3	-0.4
-0.1	-0.3	-50.5	-50.1	0.4	0.5	-49.9	-49.6	-0.1	-0.4	-50.0	-50.1
-0.5	-1.4	-50.1	-50.2	-0.6	-0.4	-100.1	-99.8	-0.5	-0.8	-200.3	-200.2
$\gamma_{(1)} = 0.070$ $\gamma_{(1),5} = 0.272$		$\gamma_{(2)} = 0.070$ $\gamma_{(2),5} = 0.268$		$\gamma_{(1)} = 0.080$ $\gamma_{(1),5} = 0.281$		$\gamma_{(2)} = 0.080$ $\gamma_{(2),5} = 0.288$		$\gamma_{(1)} = 0.103$ $\gamma_{(1),5} = 0.314$		$\gamma_{(2)} = 0.105$ $\gamma_{(2),5} = 0.312$	

Table 3. MC estimates of the point heights and SRs for disturbed observation sets

Variant A: correct order				Variant B: $h_{16}^2 = h_{16}^1$				Variant C: $h_{15}^2 = h_{15}^1$, $h_{16}^2 = h_{16}^1$			
$\hat{\mathbf{X}}_{LS,1}$	$\hat{\mathbf{X}}_{(1)}$	$\hat{\mathbf{X}}_{LS,2}$	$\hat{\mathbf{X}}_{(2)}$	$\hat{\mathbf{X}}_{LS,1}$	$\hat{\mathbf{X}}_{(1)}$	$\hat{\mathbf{X}}_{LS,2}$	$\hat{\mathbf{X}}_{(2)}$	$\hat{\mathbf{X}}_{LS,1}$	$\hat{\mathbf{X}}_{(1)}$	$\hat{\mathbf{X}}_{LS,2}$	$\hat{\mathbf{X}}_{(2)}$
0.0	2.2	0.3	-1.1	0.4	-1.5	-6.8	0.3	-0.8	0.4	-4.5	-5.2
0.4	-0.1	1.1	0.4	-0.5	-1.5	2.1	1.8	-0.2	-0.8	-5.3	-7.7
0.6	0.8	0.3	-1.5	-0.6	-3.6	3.4	1.6	-0.1	-1.0	4.9	7.4
-0.7	-0.9	0.0	1.0	0.3	-1.4	2.0	2.4	-1.3	-0.6	5.2	7.1
-0.2	0.5	-49.8	-50.3	0.4	-1.5	-36.2	-46.5	-2.0	-1.0	25.3	-42.6
$\gamma_{(1)} = 0.018$ $\gamma_{(1),5} = 0.172$		$\gamma_{(2)} = 0.019$ $\gamma_{(2),5} = 0.210$		$\gamma_{(1)} = 0.127$ $\gamma_{(1),5} = 0.321$		$\gamma_{(2)} = 0.875$ $\gamma_{(2),5} = 0.986$		$\gamma_{(1)} = 0.263$ $\gamma_{(1),5} = 0.474$		$\gamma_{(2)} = 0.887$ $\gamma_{(2),5} = 0.998$	

but not from gross errors. The natural feature of M_{split} estimation is automatic assignment of each observation to the proper epoch. Thus, we can suppose that this estimation method will not break down if such outliers occur. To illustrate such a feature of M_{split} estimation we now simulate that point P_5 is displaced and that $\Delta H_{5(1,2)} = -50$ mm. Now, let us consider the following variants of the observation sets: Variant A - both observation sets are correct (all observations are assigned to their epochs properly), Variant B – the observation h_{16} at the second epoch is equal to h_{16} at the first one, namely $h_{16}^2 = h_{16}^1$, Variant C where $h_{16}^2 = h_{16}^1$ but also $h_{15}^2 = h_{15}^1$. Thus, in Variant B and C we simulate that some observations which are assigned to the second measurement epoch should be rather related to the first one. The results which are obtained for all Variants are presented in Table 3. In the case of Variant A, the results are very close to the respective results presented in Table 1. If the observation sets are not ordered correctly, then the local SRs at the second epoch are close to 1, which means that almost always the height of the point P_5 at the second measurement epoch is better assessed by M_{split} estimation than by LS estimation. Also, the global SRs are very high at the second epoch, hence one can say that the heights of all network points are better estimated by application of M_{split} estimation.

IV. Conclusions

The paper shows that M_{split} estimation can be successfully applied in deformation analysis. The results are generally similar to the results of more conventional LS estimation; however, the latter method usually yields little bit better outcomes. The elementary tests show that the efficacy of M_{split} estimation grows with increasing shift between the observation sets. In the case of geodetic networks, where a parameter vector usually consists of several point coordinates, the shift of one or two such coordinates between measurement epochs does not influence the efficacy of M_{split} estimation in a significant way. The real advantage of M_{split} estimation is revealed for disordered observation sets, for example, when the observations from at least two measurement epochs are mixed for some reason. Note, that LS estimates break down in such a case in contrast to M_{split} estimation, for which the ordering of all observations within the combined observation set can be arbitrary and does not influence the final results of the method as well as its iterative process. Such a feature results directly from the theoretical foundations of the method which are based on the concept of the split potential. In short, each observation “chooses” the functional model which fits it best. In such a context, M_{split} estimates are robust against some kind of “outliers”, namely observations which come from other observation sets. Referring to the presented example, there are four height differences which regard the height of the network point P_5. If one of them does not fit the other, then the method tries to fit such an “outlying” observation into another epoch. If it works, then the whole estimation process succeeds. However, if such an observation is in fact affected by a gross error, then it does not fit any epoch, and the estimation must break down. The introduction of a virtual epoch, which is not related to any real measurements, would be a solution of such a problem. One can say that such an epoch can collect all “loners” which do not fit any real measurement epochs. That concept, which is by the way of out of the scope of this paper, was discussed in (Wiśniewski and Zienkiewicz, 2016; Zienkiewicz, 2014, 2018).

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